arXiv:1010.2324v1 [math.AG] 12 Oct 2010

QUIVERS, INVARIANTS AND QUOTIENT CORRESPONDENCE

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Abstract. This paper studies the geometric and algebraic aspects of the moduli spaces of quivers of fence type. We first provide two quotient presentations of the quiver varieties and interpret their equivalence as a generalized Gelfand-MacPherson correspondence. Next, we introduce parabolic quivers and extend the above from the actions of reductive groups to the actions of parabolic subgroups. Interestingly, the above geometry finds its natural counterparts in the representation theory as the branching rules and transfer principle in the context of the reciprocity algebra. The last half of the paper establishes this connection.

1. Introduction

The usual Gelfand-MacPherson correspondence ([GM82]) as formulated by Kapranov ([Ka93]) establishes a natural correspondence between GIT quotients of $\left(\mathbb{P}^{n-1}\right)^k$ by the diagonal action of $\mathrm{GL}_n(\mathbb{C})$ and GIT quotients of the Grassmannian variety $\mathrm{Gr}(n, \mathbb{C}^k)$ by the maximal torus $(\mathbb{C}^*)^k$.

In this paper, we provide two versions of the quotient correspondence for moduli spaces of quivers of fence type (§§2, 3). A quiver of fence type is a quiver $Q = (Q_0, Q_1)$ whose vertex set Q_0 can be decomposed as the disjoint union of subsets H and T such that H consists of only heads of arrows and T consists of only tails. Here Q_1 is the set of arrows. Associated to such a quiver are products of general linear groups

$$G_H = \prod_{h \in H} \mathrm{GL}_{d_h} \quad \text{and} \quad G_T = \prod_{t \in T} \mathrm{GL}_{d_t}$$

The first author was partially supported by NSA grant MSP07G-112 and NSF DMS 0901136.

where $\mathbf{d} = (d_q)_{q\in Q_0}$ is a fixed dimension vector. If for any $h \in H$ and $t \in T$, we set

$$n_h = \sum_{a\in Q_1, h(a)=h} d_{t(a)} \quad \text{and} \quad n_t = \sum_{a\in Q_1, t(a)=t} d_{h(a)},$$

then, we can associate to the quiver the following products of Grassmannians

$$X_T = \prod_{t\in T} \mathrm{Gr}(d_t, \mathbb{C}^{n_t}) \quad \text{and} \quad X_H = \prod_{h\in H} \mathrm{Gr}(d_h, \mathbb{C}^{n_h}).$$

The group G_H acts on X_T naturally; the group G_T acts on X_H naturally. For any $\mathbf{e} = (e_t)_{t\in T} \in \mathbb{N}^{|T|}$ and $\mathbf{r} = (r_h)_{h\in H} \in \mathbb{N}^{|H|}$, we have an ample line bundle

$$L_{\mathbf{e}} = \boxtimes_{t\in T} \mathcal{O}_{\mathrm{Gr}(d_t, \mathbb{C}^{n_t})}(e_t)$$

over X_T determined by $\mathbf{e}$. The tuple $\mathbf{r}$ defines a character of G_H:

$$\chi_{\mathbf{r}} : G_H \to \mathbb{C}^*$$

(see (2.2)) and thus induces a G_H-linearization $L_{\mathbf{e}}(\mathbf{r})$ over X_T. Similarly but with the roles of $\mathbf{r}$ and $\mathbf{e}$ swapped, we have an ample line bundle over X_H

$$L_{\mathbf{r}} = \boxtimes_{h\in H} \mathcal{O}_{\mathrm{Gr}(d_h, \mathbb{C}^{n_h})}(r_h),$$

a G_T-character $\chi_{\mathbf{e}} : G_T \to \mathbb{C}^*$ and the induced G_T-linearization $L_{\mathbf{r}}(\mathbf{e})$ over X_H.

Theorem 1.1. *There is a natural one-to-one correspondence between the set of GIT quotients of X_T by G_H and the set of GIT quotients of X_H by G_T. Precisely, suppose that $\mathbf{r} \in \mathbb{N}^{|H|}$ and $\mathbf{e} \in \mathbb{N}^{|T|}$ satisfy the compatibility condition* (2.3)*, then we have a natural isomorphism between $X_T^{ss}(L_{\mathbf{e}}(\mathbf{r}))/\!/G_H$ and $X_H^{ss}(L_{\mathbf{r}}(\mathbf{e}))/\!/G_T$.*

As a special case, when the quiver is a star quiver, that is, it has a unique head (H consists of a single element), we recover the quotient correspondence of [Hu05] (of which the usual GM correspondence is a special case).

The above are quotient correspondences for reductive group actions. In some practice, one may encounter quotients by parabolic groups which often requires special treatments as there is no general

quotient theory for non-reductive groups. In §3, we consider the parabolic subgroup actions on the representation space of the quiver. It turns out their quotients parameterize what we call "parabolic quivers": a parabolic quiver is a representation of the quiver Q together with some (partial) flags of V_b at every vertex $b \in Q_0$. To specify the flags, for any vertex $\nu \in Q_0$, we fix a partition

$$d_\nu = d_{\nu_1} + \cdots + d_{\nu_s}$$

of d_ν by positive integers where $s = s(\nu)$ is a positive integer depending on the vertex ν. Associated to each $\nu \in Q_0$, we have the following (partial) flag variety

$$Y_\nu := \left\{ (0 \subset V_1 \subset \cdots \subset V_s \subset \mathbb{C}^{n_\nu}) : \dim V_i = \sum_{j=1}^{i} d_{\nu_j} \right\}$$

where n_ν is as define earlier (see also (2.4)). We set

$$Y_T = \prod_{t \in T} Y_t \quad \text{and} \quad Y_H = \prod_{h \in H} Y_h.$$

We also let

$$P_H = \prod_{h \in H} P_h \quad \text{and} \quad P_T = \prod_{t \in T} P_t$$

where P_ν as the parabolic subgroup of GL_{d_ν} as defined in (3.1). Then P_H acts on Y_T naturally and P_T acts on Y_H naturally.

Theorem 1.2. *There is an one-to-one correspondence between the set of GIT quotients of* Y_T *by* P_H *and the set of GIT quotients of* Y_H *by* P_T.

This extends the quotient correspondence from the general linear groups to parabolic subgroups. For the details, see §3.

Our approaches to the above two geometric results are similar to the ones used in Theorem 4.2 of [Hu05] and also in §2.2 of [HMSV06]. For GIT quotients by parabolic subgroups, we apply the corresponding results of [Hu06].

In the second half of this paper, we turn our attention to the algebraic aspects of the above geometric results. Interestingly, our geometric correspondence finds its natural counterpart in representation theory in the context of the *reciprocity algebra* studied by Howe

and his collaborators [HL07][HTW08]. For this, we construct in §4 an algebra whose homogeneous components provide invariant section spaces as arising in the parabolic quotient correspondences. Then we show that each homogeneous component of this algebra records two different types of *branching rules* for the representations of the general linear groups. This is the algebraic version of the geometric quotient correspondence stated in the second theorem.

To be more precise, let P'_H and P'_T be the commutator subgroups of P_H and P_T respectively. Then we show that

Theorem 1.3. *The* $P'_H \times P'_T$*-invariant subring of the coordinate ring of the space* $\mathrm{Rep}(Q, \mathbf{d})$ *is a multi-graded ring whose homogeneous components encode simultaneously*

(1) *the multiplicities of the* $\prod_h \mathrm{GL}_{d_h}$ *modules* $\bigotimes_h V_h$ *in the* $\prod_{h,i} \mathrm{GL}_{d_h}$ *modules* $\bigotimes_{h,i} W_{h,i}$ *and;*
(2) *the multiplicities of the* $\prod_{h,i} \mathrm{GL}_{d_{t_{h,i}}}$ *modules* $\bigotimes_{h,i} W'_{h,i}$ *in the* $\prod_h \mathrm{GL}_{n_h}$ *modules* $\bigotimes_h V'_h$

where $h \in H$ *and* $1 \leq i \leq m(h)$. *Moreover,* V_h *and* V'_h *(respectively* $W_{h,i}$ *and* $W'_{h,i}$*) are labeled by the same Young diagrams.*

In §5, we present the parabolic quotient correspondence stated in the second theorem as a geometric version of the *transfer principle* in the representation theory.

2. Quivers and reductive quotient correspondence

2.1. A quiver is an oriented graph $Q = (Q_0, Q_1, h, t)$ equipped with a finite ordered set of vertices Q_0, a set of arrows Q_1, and two functions h, t such that for each arrow $a \in Q_1$, $h(a) \in Q_0$ is the head and $t(a) \in Q_0$ is the tail. If Q_0 contains $m > 0$ vertices, we may identify Q_0 with the set of integers $\{1, \cdots, m\}$.

2.2. Fix a vector $\mathbf{d} = (d_1, \cdots, d_m) \in \mathbb{N}^m$. The representation spaces of the quiver Q with the fixed dimension vector $\mathbf{d}$ can be, upon choosing bases of relevant vector spaces, identified with

$$\mathrm{Rep}(Q, \mathbf{d}) = \bigoplus_{a \in Q_1} \mathrm{Mat}_{d_{h(a)}, d_{t(a)}}$$

where $\mathrm{Mat}_{p,q}$ is the space of matrices of size $p \times q$. Let

$$\mathrm{GL}_{\mathbf{d}} = \prod_{i \in Q_0} \mathrm{GL}_{d_i}.$$

Then the reductive group $\mathrm{GL}_{\mathbf{d}}$ acts on $\mathrm{Rep}(Q, \mathbf{d})$ by conjugation

$$(g_1, \cdots, g_m) : (M_a)_{a \in Q_1} \to (g_{h(a)} \cdot M_a \cdot g_{t(a)}^{-1}). \tag{2.1}$$

Definition 2.3. *A quiver* $Q = (Q_0, Q_1, h, t)$ *is of fence type if there is a disjoint union* $Q_0 = H \sqcup T$ *such that* H *consists of vertices that are heads of arrows and* T *consists of vertices that are tails of arrows. Equivalently, there are no arrows between any two vertices in* H (T, *respectively*).

2.4. The reductive group $\mathrm{GL}_{\mathbf{d}} = G_H \times G_T$ acts on the affine space $\mathrm{Rep}(Q, \mathbf{d})$. Let $\mathbb{L}$ be the trivial line bundle $\mathrm{Rep}(Q, \mathbf{d}) \times \mathbb{C}$. A character

$$\chi : \mathrm{GL}_{\mathbf{d}} \longrightarrow \mathbb{C}^*$$

defines a linearization of $\mathrm{GL}_{\mathbf{d}}$ on $\mathbb{L}$ by

$$g \cdot (M, z) = (g \cdot M, \chi(g) z).$$

We can identify the center of G_H (G_T) with $(\mathbb{C}^*)^{|H|}$ ($(\mathbb{C}^*)^{|T|}$). Any character of G_H , respectively G_T, is of the form

$$\chi_{\mathbf{r}}((g_h)_{h \in H}) = \prod_{h \in H} \det(g_h)^{r_h} \text{ respectively, } \chi_{\mathbf{e}}((g_t)_{t \in T}) = \prod_{t \in T} \det(g_t)^{e_t} \tag{2.2}$$

where $\mathbf{r} = (r_h)_{h \in H} \in \mathbb{N}^{|H|}$ and $\mathbf{e} = (e_t)_{t \in T} \in \mathbb{N}^{|T|}$. The product $\chi_{\mathbf{r}} \chi_{-\mathbf{e}}$ defines a character of $\mathrm{GL}_{\mathbf{d}} = G_H \times G_T$ and any character of $\mathrm{GL}_{\mathbf{d}}$ is of this form. We let $\mathbb{L}(\chi_{\mathbf{r}} \chi_{-\mathbf{e}})$ be the associated linearized line bundle. We introduce

$$K = \left\{ (z_v I_{d_v})_{v \in Q_0} \in \mathrm{GL}_{\mathbf{d}} \mid z_v \in \mathbb{C}^*,\ z_{h(a)} = z_{t(a)},\ \forall a \in Q_1 \right\}.$$

Then the subgroup K acts trivially on $\mathrm{Rep}(Q, \mathbf{d})$. We let $\mathrm{GL}'_{\mathbf{d}} = \mathrm{GL}_{\mathbf{d}}/K$ and consider the action of this quotient group. The character $\chi_{\mathbf{r}} \chi_{-\mathbf{e}}$ descends to a character of $\mathrm{GL}'_{\mathbf{d}}$ if and only if

$$\sum_{h \in H} r_h \cdot d_h = \sum_{t \in T} e_t \cdot d_t. \tag{2.3}$$

2.5. For any $h \in H$ and $t \in T$, we set

$$n_h = \sum_{a \in Q_1, h(a)=h} d_{t(a)} \quad \text{and} \quad n_t = \sum_{a \in Q_1, t(a)=t} d_{h(a)}. \tag{2.4}$$

We also let

$$X_T = \prod_{t \in T} \mathrm{Gr}(d_t, \mathbb{C}^{n_t}) \quad \text{and} \quad X_H = \prod_{h \in H} \mathrm{Gr}(d_h, \mathbb{C}^{n_h}).$$

The group G_H acts on X_T naturally; the group G_T acts on X_H naturally. We consider the action of G_H on X_T first. We let

$$L_{\mathbf{e}} = \boxtimes_{t \in T} \mathcal{O}_{\mathrm{Gr}(d_t, \mathbb{C}^{n_t})}(e_t)$$

be the line bundle over X_T determined by $\mathbf{e}$. Then the character $\chi_{\mathbf{r}} : G_H \to \mathbb{C}^*$ of (2.2) defines a G_H-linearization $L_{\mathbf{e}}(\mathbf{r})$ over X_T. Similarly, we let

$$L_{\mathbf{r}} = \boxtimes_{h \in H} \mathcal{O}_{\mathrm{Gr}(d_h, \mathbb{C}^{n_h})}(r_h)$$

be the line bundle over X_H determined by $\mathbf{r}$. Then the character $\chi_{\mathbf{e}} : G_T \to \mathbb{C}^*$ of (2.2) defines a G_T-linearization $L_{\mathbf{r}}(\mathbf{e})$ over X_H.

Theorem 2.6. *There is a natural one-to-one correspondence between the set of GIT quotients of X_T by G_H and the set of GIT quotients of X_H by G_T. Precisely, suppose that $\mathbf{r} \in \mathbb{N}^{|H|}$ and $\mathbf{e} \in \mathbb{N}^{|T|}$ satisfy the compatibility condition* (2.3)*, then we have a natural isomorphism between $X_T^{ss}(L_{\mathbf{e}}(\mathbf{r}))/\!/G_H$ and $X_H^{ss}(L_{\mathbf{r}}(\mathbf{e}))/\!/G_T$.*

Proof. First, we can write

$$\mathrm{Rep}(Q, \mathbf{d}) = \bigoplus_{t \in T} \bigoplus_{a \in Q_1, t(a)=t} \mathrm{Mat}_{d_{h(a)}, d_t}.$$

We identify $\bigoplus_{a \in Q_1, t(a)=t} \mathrm{Mat}_{d_{h(a)}, d_t}$ with Mat_{n_t, d_t} by placing individual matrices of $\mathrm{Mat}_{d_{h(a)}, d_t}$ in rows. For simplicity, we shall write $\mathbb{L}$ for $\mathbb{L}(\chi_{\mathbf{r}} \chi_{-\mathbf{e}})$. Then by applying the first fundamental theorem of invariant theory to every individual factor $\mathrm{Gr}(d_t, \mathbb{C}^{n_t})$ of X_T, we have that for any $N \geq 0$,

$$\Gamma(X_T, L_{\mathbf{e}}^N) = \Gamma(\mathrm{Rep}(Q, \mathbf{d}), \mathbb{L}^{(\sum e_t d_t N)})^{G_T}.$$

Consequently, we have

$$\Gamma(X_T, L_{\mathbf{e}}^N)^{G_H} = \Gamma(\mathrm{Rep}(Q, \mathbf{d}), \mathbb{L}^{(\sum e_t d_t N)})^{\mathrm{GL}_{\mathbf{d}}}.$$

Likewise, we can also write

$$\mathrm{Rep}(Q,\mathbf{d}) = \bigoplus_{h\in H} \bigoplus_{a\in Q_1, h(a)=h} \mathrm{Mat}_{d_h, d_{t(a)}}.$$

We identify $\bigoplus_{a\in Q_1,h(a)=h} \mathrm{Mat}_{d_h,d_{t(a)}}$ with Mat_{d_h,n_h} by placing matrices of $\mathrm{Mat}_{d_h,d_{t(a)}}$ in different columns. Then by the first fundamental theorem of invariant theory, we obtain

$$\Gamma(X_H, L_{\mathbf{r}}^N) = \Gamma(\mathrm{Rep}(Q,\mathbf{d}), \mathbb{L}^{(\sum r_h d_h N)})^{G_H},$$

hence

$$\Gamma(X_H, L_{\mathbf{r}}^N)^{G_T} = \Gamma(\mathrm{Rep}(Q,\mathbf{d}), \mathbb{L}^{(\sum r_h d_h N)})^{\mathrm{GL}_{\mathbf{d}}}.$$

Because $\sum_{t\in T} e_t d_t = \sum_{h\in H} r_h d_h$, we see that

$$\Gamma(X_T, L_{\mathbf{e}}^N)^{G_H} = \Gamma(\mathrm{Rep}(Q,\mathbf{d}), \mathbb{L}^{(\sum r_h d_h N)})^{\mathrm{GL}_{\mathbf{d}}} = \Gamma(X_H, L_{\mathbf{r}}^N)^{G_T}.$$

This implies that we have natural isomorphisms of GIT quotients

$$X_T^{ss}(L_{\mathbf{e}}(\mathbf{r}))//G_H \cong \mathrm{Rep}(Q,\mathbf{d})^{ss}(\mathbb{L}(\chi_{\mathbf{r}}\chi_{-\mathbf{e}}))//\mathrm{GL}_{\mathbf{d}} \cong X_H^{ss}(L_{\mathbf{r}}(\mathbf{e}))//G_T.$$

□

2.7. Let $\vartheta = (\mathbf{r}, -\mathbf{e}) \in \mathbb{Z}^{|Q_0|}$. By (2.3), $\mathbf{d}\cdot\vartheta = 0$. Here, "$\cdot$" is the canonical dot product in $\mathbb{Z}^{|Q_0|}$. By King [King], a quiver representation $(V_i)_{i\in Q_0}$ is $\mathbb{L}(\chi_{\mathbf{r}}\chi_{-\mathbf{e}})$-semistable (stable) if and only if for all subrepresentation $(E_i)_{i\in Q_0}$,

$$(\dim E_i)\cdot\vartheta \le (<) 0. \tag{2.5}$$

Using this, we can give a stability criterion for the action of G_T on X_H. Let $(V_i)_{i\in H} \in X_H = \prod_{h\in H} \mathrm{Gr}(d_h, \mathbb{C}^{n_h})$ be any point. Note that for each $h \in H$, we have a fixed decomposition

$$\mathbb{C}^{n_h} = \oplus_{a\in Q_1(h)} \mathbb{C}^{d_{t(a)}}$$

where $Q_1(h) = \{a \in Q_1, h(a) = h\}$. Then $(V_i)_{i\in H}$ is semistable (stable) with respect to $L_{\mathbf{r}}(\mathbf{e})$ if and only if for all $(E_i)_{i\in H}$ with E_i a subspace of V_i, we have

$$\sum_{i\in H} r_i \dim E_i \le (<) \sum_{i\in H} \sum_{a\in Q_1(i)} e_{t(a)} \dim(E_{t(a)} \cap \mathbb{C}^{d_{t(a)}})$$

Likewise, for any point $(V_j)_{j\in T} \in X_T = \prod_{t\in T} \mathrm{Gr}(d_t, \mathbb{C}^{n_t})$, it is semistable (stable) with respect to $L_{\mathbf{e}}(\mathbf{r})$ if and only if for all $(E_j)_{j\in T}$ with E_j a subspace of V_j, we have

$$\sum_{j\in T} \sum_{a\in Q_1(j)} r_{h(a)} \dim(E_{h(a)} \cap \mathbb{C}^{d_{h(a)}}) \leq (<) \sum_{j\in T} e_j \dim E_j$$

2.8. Note that either of the GIT quotients $X^{ss}(L_{\mathbf{e}}(\mathbf{r}))/\!/G_H$ and $Y^{ss}(L_{\mathbf{r}}(\mathbf{e}))/\!/G_T$ is the quiver moduli $\mathcal{R}ep(Q, \mathbf{d})_\vartheta$ determined by $\vartheta = (\mathbf{r}, -\mathbf{e})$ for the fence quiver (2.3) with the given dimension vector $\mathbf{d}$. It would be an interesting problem to find other classes of quivers and dimension vectors such that their quiver varieties have the kind of geometric interpretations as in Theorem 2.6.

2.9. As examples, we now revisit the GM correspondence of [Hu05] using the language of quivers. For this, we let Q be the star quiver with vertices $Q_0 = \{0, 1, \cdots, m\}$ with

$$Q_1 = \{1 \to 0, \cdots, m \to 0\}$$

(0 is the unique head for all arrows). This is a special case of fence quivers where H consists of a single vertex. Let

$$\mathbf{d} = (n, k_1, \cdots, k_m) \in \mathbb{N}^{m+1}$$

such that for each $1 \leq i \leq m$, $k_i \leq n \leq \sum_i k_i$. In this case, we have

$$\mathrm{Rep}(Q, \mathbf{d}) = \bigoplus_{i=1}^{m} \mathrm{Mat}_{n,k_i}, \quad \mathrm{GL}_{\mathbf{d}} = \mathrm{GL}_n \times \prod_{i=1}^{m} \mathrm{GL}_{k_i}$$

where as in (2.1), GL_n acts by left multiplication and $\prod_{i=1}^m \mathrm{GL}_{k_i}$ acts on the right by the inverse multiplication, component-wise. We have in this case

$$X_T = \prod_{i=1}^{m} \mathrm{Gr}(k_i, \mathbb{C}^n) \quad \text{and} \quad X_H = \mathrm{Gr}(n, \mathbb{C}^k)$$

where $k = \sum_{i=1}^m k_i$. The group GL_n acts on X_T naturally and the group $\prod_{i=1}^m \mathrm{GL}_{k_i}$ acts on X_H in the natural way. Let $r \in \mathbb{N}$ and $\mathbf{e} \in \mathbb{N}^m$ such that

$$r \cdot n = \sum_i e_i \cdot k_i. \tag{2.6}$$

Then as a special case of Theorem 2.10 , we have

Corollary 2.10. ([Hu05]) *There is a natural one-to-one correspondence between the set of GIT quotients of* $\prod_{i=1}^m \mathrm{Gr}(k_i, \mathbb{C}^n)$ *by* GL_n *and the set of GIT quotients of* $\mathrm{Gr}(n, \mathbb{C}^k)$ *by* $\prod_{i=1}^m \mathrm{GL}_{k_i}$*. Precisely, suppose that* $\mathbf{r} \in \mathbb{N}$ *and* $\mathbf{e} \in \mathbb{N}^m$ *satisfy the compatibility condition* $rn = \sum_i e_i k_i$*, then we have a natural isomorphism between* $\prod_{i=1}^m \mathrm{Gr}(k_i, \mathbb{C}^n)(L_{\mathbf{e}}(\mathbf{r}))//\mathrm{GL}_n$ *and* $\mathrm{Gr}(n, \mathbb{C}^k)^{ss}(L_{\mathbf{r}}(\mathbf{e}))// \prod_{i=1}^m \mathrm{GL}_{k_i}$*.*

3. Parabolic Quivers and Correspondence

Definition 3.1. *Let* $Q = (Q_0, Q_1, h, t)$ *be a quiver. A representation of* Q *with parabolic structures is a representation of the quiver* Q *together with some (partial) flags of* V_b *at every vertex* $b \in Q_0$*.*

3.2. At some vertices, the partial flags may be trivial. When all are trivial, we have an ordinary quiver representation.

3.3. To specify the flags, for any vertex $v \in Q_0$, we fix a partition

$$d_v = d_{v_1} + \cdots + d_{v_s}$$

of d_v by positive integers where $s = s(v)$ is a positive integer depending on the vertex v. We let P_v be the parabolic subgroup of GL_{d_v} consisting of the block upper triangular matrices whose diagonal blocks are of the size $(d_{v_1}, \cdots, d_{v_s})$

$$P_v = \left\{ \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1s} \\ & B_{22} & & \vdots \\ & & \ddots & \\ 0 & & & B_{ss} \end{bmatrix} \right\} \tag{3.1}$$

where $B_{ii} \in \mathrm{GL}_{d_{v_i}}$ for $1 \le i \le s$.

3.4. Associated to each $v \in Q_0$, we define the following (partial) flag variety

$$Y_v := \mathrm{Fl}(d_{v_1}, \cdots, d_{v_s}; \mathbb{C}^{n_v}) = \left\{ (0 \subset V_1 \subset \cdots \subset V_s \subset \mathbb{C}^{n_v}) : \dim V_i = \sum_{j=1}^{i} d_{v_j} \right\}$$

where n_ν is as define in (2.4). For every $\nu \in Q_0$, an s-tuple $\mathbf{r} = (r_1, \cdots, r_s) \in \mathbb{N}^s$ defines a (very) ample line bundle over Y_ν:

$$L_{\mathbf{r}} = \mathcal{O}_{Gr(d_{\nu_1}, \mathbb{C}^{n_\nu})}(r_1) \otimes \mathcal{O}_{Gr(d_{\nu_1}+d_{\nu_2}, \mathbb{C}^{n_\nu})}(r_2) \otimes \cdots \otimes \mathcal{O}_{Gr(d_\nu, \mathbb{C}^{n_\nu})}(r_s)$$

(recall here that $s = s(\nu)$ depends on ν). This line bundle is induced from the Plücker embedding of the flag variety Y_ν into the product of the Grassmannian $\prod_i Gr(\sum_{j=1}^i d_{\nu_j}, \mathbb{C}^{n_\nu})$. The action of the reductive group GL_{d_ν} lifts canonically to $L_{\mathbf{r}}$, making it a GL_{d_ν}-linearized line bundle. The parabolic subgroup P_ν inherits the action.

3.5. The parabolic subgroup P_ν has the group of characters of dimension s, we can twist the P_ν-linearized line bundle $L_{\mathbf{r}}$ by its characters. For this, note that the commutator subgroup P'_ν of P_ν consists of elements with $B_{ii} \in SL_{d_{\nu_i}}$ for each i as in (3.1). Therefore, we have $P_\nu/P'_\nu \cong (\mathbb{C}^*)^s$ and this can be identified with the center of P_ν by

$$(\tau_1, \cdots, \tau_s) \longmapsto \begin{bmatrix} \tau_1 I_{d_{\nu_1}} & & & 0 \\ & \tau_2 I_{d_{\nu_2}} & & \\ & & \ddots & \\ 0 & & & \tau_s I_{d_{\nu_s}} \end{bmatrix} \in P_\nu. \tag{3.2}$$

Now for any s-tuple of positive integers

$$\mathbf{e} = (e_1, \cdots, e_s) \in \mathbb{N}^s,$$

it defines a character $\mu_{\mathbf{e}}$ of P_ν

$$\begin{aligned} \mu_{\mathbf{e}} : P_\nu &\to \mathbb{C}^* \\ \mu_{\mathbf{e}}(\tau_1, \cdots, \tau_s) &= \tau_1^{d_{\nu_1}(e_1+e_2+\cdots+e_s)} \tau_2^{d_{\nu_2}(e_2+e_3+\cdots+e_s)} \cdots \tau_s^{d_{\nu_s} e_s}. \end{aligned} \tag{3.3}$$

Then the character defines a P_ν-linearized line bundle $L_{\mathbf{r}}(\mathbf{e})$ over Y_ν. We should make a useful note here: both $\mathbf{r}$ and $\mathbf{e}$ are some chosen s-tuples of positive integers with $s = s(\nu)$ depending on ν; $\mathbf{r}$ defines a line bundle $L_{\mathbf{r}}$; $\mathbf{e}$ defines a character $\mu_{\mathbf{e}}$. Of course, the roles of $\mathbf{r}$ and $\mathbf{e}$ can be switched.

3.6. Now we set

$$Y_T = \prod_{t \in T} Y_t \quad \text{and} \quad Y_H = \prod_{h \in H} Y_h.$$

We also let

$$P_H = \prod_{h \in H} P_h \quad \text{and} \quad P_T = \prod_{t \in T} P_t.$$

Then P_H acts on Y_T naturally and P_T acts on Y_H naturally. Suppose that for every $\nu \in Q_0$, we have chosen a pair of $s(\nu)$-tuples $\mathbf{r}(\nu)$ and $\mathbf{e}(\nu)$ in $\mathbb{N}^{s(\nu)}$. Then over Y_T, we have an induced ample line bundle

$$L_{\mathbf{r}_T} = \boxtimes_{t \in T} L_{\mathbf{r}(t)}.$$

For any $h \in H$, $\mathbf{e}(h)$ defines a character of P_h, hence their product induces a character $\mathbf{e}_H$ of P_H. This gives rise to a P_H-linearized line bundle $L_{\mathbf{r}_T}(\mathbf{e}_H)$. Thus we have the Zariski open subset $Y_T^{ss}(L_{\mathbf{r}_T}(\mathbf{e}_H))$ and its quotient $Y_T^{ss}(L_{\mathbf{r}_T}(\mathbf{e}_H))//P_H$ (see [Hu06] for a quotient theory of parabolic subgroup actions).

Likewise, we have the induced ample line bundle

$$L_{\mathbf{e}_H} = \boxtimes_{h \in H} L_{\mathbf{e}(h)}$$

over Y_H. Then the product of all characters $\mathbf{r}(t)$ for all $t \in T$ gives rise to a character $\mathbf{r}_T$ of P_T and hence a P_T-linearized line bundle $L_{\mathbf{e}_H}(\mathbf{r}_T)$. Thus we have the Zariski open subset $Y_H^{ss}(L_{\mathbf{e}_H}(\mathbf{r}_T))$ and its quotient $Y_H^{ss}(L_{\mathbf{e}_H}(\mathbf{r}_T))//P_T$.

3.7. We are now almost ready to state our main geometric theorem of this section. Before making the statement, for the similar reason as in (2.3), we need to impose a condition.

$$(3.4)\quad \sum_{t \in T} \sum_{1 \le i \le s(t)} d_{t_i}(r_i + \cdots + r_{s(t)}) = \sum_{h \in H} \sum_{1 \le i \le s(h)} d_{h_i}(e_i + \cdots + e_{s(h)}).$$

Theorem 3.8. *There is an one-to-one correspondence between the set of GIT quotients of Y_T by P_H and the set of GIT quotients of Y_H by P_T. More precisely, assume that for all $\nu \in Q_0$, the chosen pairs $\mathbf{r}(\nu)$ and $\mathbf{e}(\nu)$ in $\mathbb{N}^{s(\nu)}$ satisfy (3.4). Then we have a natural isomorphism between the quotient $Y_T^{ss}(L_{\mathbf{r}_T}(\mathbf{e}_H))//P_H$ and the quotient $Y_H^{ss}(L_{\mathbf{e}_H}(\mathbf{r}_T))//P_T$.*

Proof. As in the proof of Theorem 2.6, we first identify $\mathrm{Rep}(Q, \mathbf{d})$ with

$$\bigoplus_{t \in T} \mathrm{Mat}_{n_t, d_t}.$$

Then for any $N > 0$, by [Fu97, §9], we have

$$\Gamma(Y_T, L_{\mathbf{r}_T}^{\otimes N}) \cong \Gamma(\mathrm{Rep}(Q, \mathbf{d}), \mathbb{L}^{\otimes bN})^{P_T}$$

where $b = \sum_{t\in T}\sum_{1\le i\le s(t)} d_{t_i}(r_i + \cdots + r_{s(t)})$. Likewise, we have

$$\Gamma(Y_H, L_{\mathbf{r}_H}^{\otimes N}) \cong \Gamma(\mathrm{Rep}(Q, \mathbf{d}), \mathbb{L}^{\otimes cN})^{P_H}$$

$c = \sum_{h\in H}\sum_{1\le i\le s(h)} d_{h_i}(r_i + \cdots + r_{s(h)})$. Note that $b = c$. Hence, we have

$$\Gamma(Y_T, L_{\mathbf{r}_T}(\mathbf{e}_H)^{\otimes N})^{P_H} \cong \Gamma(\mathrm{Rep}(Q, \mathbf{d}), \mathbb{L}^{\otimes bN})^{P_H\times P_T} \cong \Gamma(Y_H, L_{\mathbf{r}_H}^{\otimes N})^{P_T}.$$

This implies the natural isomorphisms of the quotients

$$Y_T^{ss}(L_{\mathbf{r}_T}(\mathbf{e}_H))//P_H \cong \mathrm{Rep}(Q, \mathbf{d})^{ss}(\mathbf{r}, \mathbf{e})//(P_H \times P_T) \cong Y_H^{ss}(L_{\mathbf{e}_H}(\mathbf{r}_T))//P_T.$$

$\square$

Remark 3.9. Note that the quotient $\mathrm{Rep}(Q, \mathbf{d})^{ss}(\mathbf{r}, \mathbf{e})//(P_H \times P_T)$ parameterizes the equivalence classes of semistable parabolic quivers.

3.10. As a special case, here we assume that all the parabolic structures on tails are trivial. In such a case, Theorem 3.8 will specialize to a correspondence between quotients of a reductive group action and quotients of a parabolic subgroup. This gives a GM Correspondence of mixed types. To be more precise, in this case, we have $Y_T = X_T = \prod_{t\in T} \mathrm{Gr}(d_t, \mathbb{C}^{n_t})$ is a product of Grassmannians and $Y_H = \prod_{h\in H} Y_h$ is a product of flag varieties. Acting on X_T is the parabolic group P_H; acting on Y_H is the reductive group G_T. The condition (3.4) becomes

$$\sum_{t\in T} d_t r_t = \sum_{h\in H}\sum_{1\le i\le s(h)} d_{h_i}(e_i + \cdots + e_{s(h)}). \tag{3.5}$$

Under this condition, we have a natural isomorphism between the quotient $X_T^{ss}(L_{\mathbf{r}_T}(\mathbf{e}_H))//P_H$ by parabolic subgroup and the quotient $Y_H^{ss}(L_{\mathbf{e}_H}(\mathbf{r}_T)//G_T$ by reductive group. (For further correspondence between quotients by a reductive group and quotients by its parabolic subgroups, consult [Hu06].)

3.11. We may further specialize to the case when Q is the star quiver as considered in 2.9. In this case, Y_T is $\prod_{t\in T}\mathrm{Gr}(d_t,\mathbb{C}^n)$ and Y_H is the (single, partial) flag variety consisting flags of the type

$$0\subset V_1\subset\cdots\subset V_s\subset\mathbb{C}^d,\quad \dim V_i=\sum_{j=1}^{i}n_j$$

where $d=\sum_{t\in T}d_t$ and $n=\sum_{i=1}^{s}n_i$ is a partition of n. Over the product of the Grassmannians Y_T, we have a natural diagonal action of the parabolic group P_H; over the (partial) flag variety Y_H, we have the action of $G_T=\prod \mathrm{GL}_{d_t}$. The condition (3.4) now reads

$$\sum_{t\in T}d_t r_t=\sum_i n_i(e_i+\cdots+e_s). \tag{3.6}$$

4. Multi-Reciprocity Algebras

In this section, we study representation theoretic correspondences matching with our geometric ones.

4.1. For reductive groups K and G with $K\subset G$, we consider irreducible representations V and W of K and G, respectively. By Schur's lemma, the multiplicity of V in W as a representation of K is equal to the dimension of the space

$$\mathrm{Hom}_K(V,W),$$

which is called the *multiplicity space*. The *branching rule* under the restriction of G down to K is a description of the multiplicity spaces. A special case of the above is when $G=K\times\cdots\times K$ and K is identified with the diagonal subgroup in G. In this case, the multiplicity spaces

$$\mathrm{Hom}_K(V,W_1\otimes\cdots\otimes W_m)$$

describe the decompositions of tensor products of K-modules W_i. In what follows, we shall describe the invariant section spaces

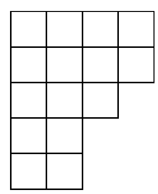
$$\Gamma(\mathrm{Rep}(Q,\mathbf{d}),\mathbb{L}^{\otimes b})^{P_H\times P_T}$$

as graded components of an algebra encoding branching rules of different types.

4.2. First we recall Young diagrams as a labeling system for irreducible polynomial representations of GL_n. Every irreducible polynomial representation of GL_n is uniquely labeled by, under the identification with its highest weight, a Young diagram with no more than n rows. Let ρ_n^F denote the irreducible representation of GL_n labeled by Young diagram $F = (f_1, \cdots, f_n) \in \mathbb{Z}^n$ with $f_1 \geq \cdots \geq f_n \geq 0$. Here, $(f_1, \cdots, f_n)$ is identified with the highest weight of ρ_n^F . The dual representation of ρ_n^F has the highest weight

$$F^* = (-f_n, \cdots, -f_1)$$

and will be denoted by $\rho_n^{F^*}$. We write $\ell(F)$ for the number of nonzero entries in F. If the entries f_i of F repeat a_i times, then we also write $F = (f_1^{a_1}, f_2^{a_2}, \cdots)$ with $f_1 > f_2 > \cdots > 0$. For example, $F = (4, 4, 3, 2, 2)$ or $(4^2, 3^1, 2^2)$ can be drawn as



and $\ell(F) = 5$. Then the Young diagram for F^* can be drawn by rotating F around its center by $180°$.

4.3. For $(k_1, \cdots, k_s) \in \mathbb{N}^s$, let $k = k_1 + \cdots + k_s$. We let P_k denote the parabolic subgroup of GL_k consisting of the block upper triangular matrices whose diagonal blocks are of the sizes $k_1, \cdots, k_s$

$$P_k = \left\{ \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1s} \\ & B_{22} & & \vdots \\ & & \ddots & \\ 0 & & & B_{ss} \end{bmatrix} \right\}$$

where $B_{ii} \in GL_{k_i}$ for $1 \leq i \leq s$.

Lemma 4.4. *Let P_k be the parabolic subgroup of GL_k as given above and P'_k be its commutator subgroup. Then the dimension of the P'_k-invariant subspace of ρ_k^F is at most 1. It equals 1 if and only if F is of the form*

$$(f_1^{k_1}, f_2^{k_2}, \cdots, f_s^{k_s}) \in \mathbb{N}^s$$

with $f_1 > \cdots > f_s > 0$*. In particular, the dimension of* $(\rho_k^F)^{SL_k}$ *is* 1 *only when* F *is of the form* (f_1^k)*, i.e., a rectangular diagram with* k *rows.*

Proof. Note that P'_k contains the maximal unipotent subgroup U_k of GL_k. By highest weight theory (e.g., [GW09, §3]), $(\rho_k^F)^{U_k}$ is the one dimensional subspace spanned by a highest weight vector of ρ_k^F. Therefore, the dimension of $(\rho_k^F)^{P'_k}$ is less than or equal to 1. The condition for this being exactly 1 can be obtained by simply computing invariants under the diagonal blocks of P'_k or can be found in [Fu97, §9.3]. ☐

The same statement holds also for the dual representations $\rho_k^{F^*}$ and F^*.

4.5. As we noted in (3.2) we have $P_k/P'_k \cong (\mathbb{C}^*)^s$. Then for Young diagram $F = (f_1^{k_1}, \cdots, f_s^{k_s})$, P_k/P'_k is acting on the one dimensional space $(\rho_k^F)^{P'_k}$ via the character

$$\mu_F : (\mathbb{C}^*)^s \to \mathbb{C}^*$$
$$\mu_F(\tau_1, \cdots, \tau_s) = \tau_1^{k_1 f_1} \cdots \tau_s^{k_s f_s}$$

This notation is the same as our previous one $\mu_{\mathbf{e}}$ for $\mathbf{e} = (e_1, \cdots, e_s) \in \mathbb{N}^s$ given in (3.3) by setting $f_i = e_i + \cdots + e_s$ for $1 \le i \le s$. We let A_k^+ denote the semigroup of the characters of P_k

$$A_k^+ = \{\mu_F : F = (f_1^{k_1}, \cdots, f_s^{k_s})\}.$$

4.6. Now we give an action of $GL_n \times GL_k$ on the space $\mathrm{Mat}_{n,k}$ by

$$(g_1, g_2) \cdot M = g_1 M g_2^{-1}$$

for $(g_1, g_2) \in GL_n \times GL_k$ and $M \in \mathrm{Mat}_{n,k}$.

Lemma 4.7. (1) *With respect to the action of* $GL_n \times GL_k$*, the coordinate algebra* $\mathbb{C}[\mathrm{Mat}_{n,k}]$ *of* $\mathrm{Mat}_{n,k}$ *decomposes as*

$$\mathbb{C}[\mathrm{Mat}_{n,k}] = \sum_{\ell(F) \le \min(n,k)} \rho_n^{F^*} \otimes \rho_k^F$$

where the sum runs over all F *with less than or equal to* $\min(n, k)$ *rows.*

(2) *For a parabolic subgroup* P_k *of* GL_k*, the* P'_k*-invariant subalgebra of* $\mathbb{C}[\mathrm{Mat}_{n,k}]$ *is graded by the semigroup* A_k^+ *and has decomposition*

$$\mathbb{C}[\mathrm{Mat}_{n,k}]^{P'_k} = \sum_F \rho_n^{F^*}$$

over F *such that* $\ell(F) \leq \min(n,k)$ *and* $\dim\left(\rho_k^F\right)^{P'_k} = 1$.

Proof. The first statement is known as the GL_n-GL_k duality (e.g., [GW09, Theorem 5.6.7]). For the second statement, by taking $(1 \times P'_k)$-invariant subalgebra of $\mathbb{C}[\mathrm{Mat}_{n,k}]$, we have

$$\mathbb{C}[\mathrm{Mat}_{nk}]^{P'_k} = \sum_{\ell(F)\leq\min(n,k)} \rho_n^{F^*} \otimes \left(\rho_k^F\right)^{P'_k}$$

Then for each F, from Lemma 4.4, the space $\rho_n^{F^*} \otimes \left(\rho_k^F\right)^{P'_k}$ is nonzero exactly when $\dim\left(\rho_k^F\right)^{P'_k} = 1$ and in this case the space

$$\rho_n^{F^*} \otimes \left(\rho_k^F\right)^{P'_k} \cong \rho_n^{F^*}$$

is the A_k^+-eigenspace with weight μ_F. Hence $\mathbb{C}[\mathrm{Mat}_{nk}]^{P'_k}$ is the sum of A_k^+-eigenspaces, and this gives the grading structure. □

4.8. Let us begin with the coordinate algebra $\mathbb{C}[\mathrm{Rep}(Q,\mathbf{d})]$ of the representation space of a fence quiver $Q = (Q_0, Q_1, h, t)$ with the dimension vector $\mathbf{d}$. We shall use the same notation introduced in Section 3.

By using the identification of $\mathrm{Rep}(Q,\mathbf{d})$ with the direct sum of the spaces $\mathrm{Mat}_{d_{h(a)},d_{t(a)}}$, we have

$$\begin{aligned} \mathbb{C}[\mathrm{Rep}(Q,\mathbf{d})] &= \bigotimes_{a\in Q_1} \mathbb{C}[\mathrm{Mat}_{d_{h(a)},d_{t(a)}}] \\ &= \bigotimes_{h\in H} \bigotimes_{a\in Q_1(h)} \mathbb{C}[\mathrm{Mat}_{d_h,d_{t(a)}}] \\ &= \bigotimes_{h\in H} \mathbb{C}[\mathrm{Mat}_{d_h,n_h}] \end{aligned} \tag{4.1}$$

where $Q_1(h) = \{a \in Q_1 : h(a) = h\}$ and $n_h = \sum_{a\in Q_1(h)} d_{t(a)}$.

To be more precise, for any fixed $h \in H$, we set

$$\{t(a) : a \in Q_1(h)\} = \left\{t_{h,1}, \cdots, t_{h,m(h)}\right\}.$$

We shall show that for each $h \in H$, the $(P'_{d_h} \times \prod P'_{d_{t_{h,i}}})$-invariant subalgebra of the algebra $\mathbb{C}[\mathrm{Mat}_{d_h,n_h}]$ records two different types of branching rules with respect to the following restrictions:

$$\begin{aligned} &\mathrm{GL}_{d_h} \subset \mathrm{GL}_{d_h} \times \cdots \times \mathrm{GL}_{d_h} \ (m(h) \text{ copies}); \\ &\mathrm{GL}_{d_{t_{h,1}}} \times \cdots \times \mathrm{GL}_{d_{t_{h,m(h)}}} \subset \mathrm{GL}_{n_h}. \end{aligned}$$

Then by iterating this result over $h \in H$, we can obtain a complete description of $(P'_H \times P'_T)$-invariant subalgebra of $\mathbb{C}[\mathrm{Rep}(Q, \mathbf{d})]$ to show the following theorem.

Theorem 4.9. (1) *The algebra* $\mathbb{C}[\mathrm{Rep}(Q, \mathbf{d})]^{P'_H \times P'_T}$ *is graded by*

$$\prod_h \left(A^+_{d_h} \times \prod_{i=1}^{m(h)} A^+_{d_{t_{h,i}}} \right).$$

(2) *For each* $h \in H$*, we consider Young diagrams* $F(h)$ *and* $D(h, i)$ *such that*

$$\dim \left(\rho^{F(h)}_{d_h} \right)^{P'_{d_h}} = \dim \left(\rho^{D(h,i)}_{d_{t_{h,i}}} \right)^{P'_{d_{t_{h,i}}}} = 1$$

for $1 \leq i \leq m(h)$*.*

The dimension of the $\prod_h (\mu_{F(h)}, \mu_{D(h,1)}, \cdots, \mu_{D(h,m(h))})$*-homogeneous component for the algebra* $\mathbb{C}[\mathrm{Rep}(Q, \mathbf{d})]^{P'_H \times P'_T}$ *records simultaneously*

(a) *the multiplicity of the* $\prod_h \mathrm{GL}_{d_h}$*-module* $\bigotimes_h \rho^{F(h)}_{d_h}$ *in*

$$\bigotimes_h \left(\rho^{D(h,1)}_{d_h} \otimes \cdots \otimes \rho^{D(h,m(h))}_{d_h} \right);$$

(b) *the multiplicity of the* $\prod_h \prod_i \mathrm{GL}_{d_{t_{h,i}}}$*-module*

$$\bigotimes_h \left(\rho^{D(h,1)}_{d_{t_{h,1}}} \otimes \cdots \otimes \rho^{D(h,m(h))}_{d_{t_{h,m(h)}}} \right)$$

in $\bigotimes_h \rho^{F(h)}_{n_h}$*.*

To prove this theorem, we investigate the individual components $\mathbb{C}[\mathrm{Mat}_{d_h,n_h}]$ of $\mathbb{C}[\mathrm{Rep}(Q, \mathbf{d})]$ given in (4.1). Then the theorem can be obtained simply by repeating Corollary 4.16 on $\mathbb{C}[\mathrm{Mat}_{d_h,n_h}]$ over $h \in H$.

4.10. To simplify our notation, we write n for d_h and c_i for $d_{t_{h,i}}$ for each i. Also, let $\underline{c} = (c_1, \cdots, c_m)$ and $c = c_1 + \cdots + c_m$, and set

$$\begin{aligned} GL_{\underline{c}} &= \prod_i GL_{c_i} \quad \text{and} \quad GL_{\underline{n}} = \prod GL_n \ (m \text{ copies}) \\ P'_{\underline{c}} &= \prod_i P'_{c_i} \quad \text{and} \quad A^+_{\underline{c}} = \prod_i A^+_{c_i} \end{aligned}$$

Proposition 4.11. *The following is an* $\left(A^+_n \times A^+_{\underline{c}}\right)$*-graded algebra decomposition of* $\mathbb{C}[\mathrm{Mat}_{nk}]^{P'_n \times P'_{\underline{c}}}$

$$\sum_{(D_1,\cdots,D_m)} \sum_F \mathrm{Hom}_{GL_n}(\rho^F_n, \rho^{D_1}_n \otimes \cdots \otimes \rho^{D_m}_n) \otimes \left(\rho^{F^*}_n\right)^{P'_n}$$

where the sum runs over F *and* D_i *such that* $\ell(F) \leq \min(n, c)$, $\ell(D_i) \leq \min(n, c_i)$, *and*

$$\dim\left(\rho^F_n\right)^{P'_n} = \dim\left(\rho^{D_i}_{c_i}\right)^{P'_{c_i}} = 1$$

for $1 \leq i \leq m$. *Each graded component tells us how a* $GL_{\underline{n}}$*-irreducible representation decomposes as a* GL_n*-module.*

Proof. By repeating the GL_n-GL_{c_i} dualities to the blocks of $\mathrm{Mat}_{n,c} = \mathrm{Mat}_{n,c_1} \oplus \cdots \oplus \mathrm{Mat}_{n,c_m}$, we obtain

$$\begin{aligned} \mathbb{C}[\mathrm{Mat}_{n,c}] &= \mathbb{C}[\mathrm{Mat}_{n,c_1}] \otimes \cdots \otimes \mathbb{C}[\mathrm{Mat}_{n,c_m}] \\ &= \sum_{(D_1,\cdots,D_m)} \left(\rho^{D_1^*}_n \otimes \rho^{D_1}_{c_1}\right) \otimes \cdots \otimes \left(\rho^{D_m^*}_n \otimes \rho^{D_m}_{c_m}\right) \\ &= \sum_{(D_1,\cdots,D_m)} \left(\rho^{D_1^*}_n \otimes \cdots \otimes \rho^{D_m^*}_n\right) \otimes \left(\rho^{D_1}_{c_1} \otimes \cdots \otimes \rho^{D_m}_{c_m}\right) \end{aligned}$$

where the sum runs over all m-tuples $(D_1, \cdots, D_m)$ with $\ell(D_i) \leq \min(n, c_i)$ for each i. Then the $P'_{\underline{c}}$-invariants give us

$$\mathbb{C}[\mathrm{Mat}_{n,c}]^{P'_{\underline{c}}} = \sum_{(D_1,\cdots,D_m)} \left(\rho^{D_1^*}_n \otimes \cdots \otimes \rho^{D_m^*}_n\right) \otimes \left(\rho^{D_1}_{c_1}\right)^{P'_{c_1}} \otimes \cdots \otimes \left(\rho^{D_m}_{c_m}\right)^{P'_{c_m}}.$$

Note that by Lemma 4.4, the dimension of

$$W_{(D_1,\cdots,D_m)} = \left(\rho^{D_1}_{c_1}\right)^{P'_{c_1}} \otimes \cdots \otimes \left(\rho^{D_m}_{c_m}\right)^{P'_{c_m}}$$

is at most 1. Then the invariant algebra $\mathbb{C}[\mathrm{Mat}_{nc}]^{P'_{\underline{c}}}$ is graded by the semigroup $A^+_{\underline{c}}$ or the set of these m-tuples $(D_1, \cdots, D_m)$ of Young

diagrams, and its graded components are exactly m-fold tensor products

$$V_{(D_1,\cdots,D_m)} = \rho_n^{D_1^*} \otimes \cdots \otimes \rho_n^{D_m^*}$$

of irreducible representations of GL_n.

$V_{(D_1,\cdots,D_m)}$, as a representation of GL_n, can be decomposed as

$$\begin{aligned} V_{(D_1,\cdots,D_m)} &= \rho_n^{D_1^*} \otimes \cdots \otimes \rho_n^{D_m^*} \\ &= \sum_{(F_2,\cdots,F_m)} c_{D_1D_2}^{F_2} c_{F_2D_3}^{F_3} \cdots c_{F_{m-1}D_m}^{F_m} \left(\rho_n^{F^*}\right) \end{aligned}$$

where $F = F_m$ and $c_{F_{i-1}D_i}^{F_i}$ is the Littlewood-Richardson number, i.e., the multiplicity of $\rho_n^{F_i}$ in $\rho_n^{F_{i-1}} \otimes \rho_n^{D_i}$ with the convention $D_1 = F_1$. Therefore, $V_{(D_1,\cdots,D_m)}$ contains

$$\sum_{(F_2,\cdots,F_m)} c_{D_1D_2}^{F_2} c_{F_2D_3}^{F_3} \cdots c_{F_{m-1}D_m}^{F_m}$$

copies of $\rho_n^{F^*}$. Note that if $\ell(F_i) > \min(n, \ell(F_{i-1}) + \ell(D_i))$, then $c_{F_{i-1}D_i}^{F_i} = 0$ for all i. Therefore, in particular, $\ell(F)$ should be less than or equal to $\min(n, c)$.

For F with $\ell(F) \le \min(n, c)$ and $\dim\left(\rho_n^F\right)^{P'_n} = 1$, this multiplicity is equal to the dimension of the invariant space

$$\begin{aligned} \left(V_{(D_1,\cdots,D_m)}\right)^{P'_n} &= \sum_{(F_2,\cdots,F_m)} c_{D_1D_2}^{F_2} c_{F_2D_3}^{F_3} \cdots c_{F_{m-1}D_m}^{F_m} \left(\rho_n^{F^*}\right)^{P'_n} \\ &\cong \sum_F \mathrm{Hom}_{GL_n}\left(\rho_n^F, \rho_n^{D_1} \otimes \cdots \otimes \rho_n^{D_m}\right) \otimes \left(\rho_n^{F^*}\right)^{P'_n} \end{aligned}$$

and we see that the P'_n-invariant algebra of $\mathbb{C}[\mathrm{Mat}_{n,c}]^{P'_{\underline{c}}}$ has the decomposition

$$\begin{aligned} &\sum_{(D_1,\cdots,D_m)} \left(V_{(D_1,\cdots,D_m)}\right)^{P'_n} \otimes W_{(D_1,\cdots,D_m)} \\ \cong\ & \sum_{(D_1,\cdots,D_m)} \sum_F \left(V_{(D_1,\cdots,D_m)}\right)^{P'_n} \\ \cong\ & \sum_{(D_1,\cdots,D_m)} \sum_F \mathrm{Hom}_{GL_n}\left(\rho_n^F, \rho_n^{D_1} \otimes \cdots \otimes \rho_n^{D_m}\right) \otimes \left(\rho_n^{F^*}\right)^{P'_n} \end{aligned}$$

and each graded component is an $\left(A_n^+ \times A_{\underline{c}}^+\right)$-eigenspace. Consequently, the graded components of the invariant algebra $\mathbb{C}[\mathrm{Mat}_{n,c}]^{P'_n \times P'_{\underline{c}}}$ describe the branching multiplicities under the restrictions of $\mathrm{GL}_{\underline{n}}$ down to the diagonal GL_n. $\square$

4.12. Next, in taking the invariants of $(P'_n \times P'_{\underline{c}})$, by reversing the order of the procedures, we consider the invariants of P'_n first. This provides us a different representation theoretic description of the $(P'_n \times P'_{\underline{c}})$-invariant subalgebra of $\mathbb{C}[\mathrm{Mat}_{n,c}]$.

Proposition 4.13. *The following is an* $\left(A_n^+ \times A_{\underline{c}}^+\right)$*-graded algebra decomposition of* $\mathbb{C}[\mathrm{Mat}_{nk}]^{P'_n \times P'_{\underline{c}}}$

$$\sum_F \sum_{(D_1,\cdots,D_m)} \mathrm{Hom}_{\mathrm{GL}_{\underline{c}}}\left(\rho_{c_1}^{D_1} \otimes \cdots \otimes \rho_{c_m}^{D_m}, \rho_c^F\right) \otimes \left(\rho_{c_1}^{D_1}\right)^{P'_{c_1}} \otimes \cdots \otimes \left(\rho_{c_m}^{D_m}\right)^{P'_{c_m}}$$

where the sum runs over F *and* D_i *such that* $\ell(F) \leq \min(n, c)$, $\ell(D_i) \leq \min(n, c_i)$*, and*

$$\dim\left(\rho_n^F\right)^{P'_n} = \dim\left(\rho_{c_i}^{D_i}\right)^{P'_{c_i}} = 1$$

for $1 \leq i \leq m$*. Each graded component tells us how a* GL_c *irreducible representation decomposes as a* $\mathrm{GL}_{\underline{c}}$*-module.*

Proof. Starting from the GL_n-GL_c duality, we have

$$\mathbb{C}[\mathrm{Mat}_{n,c}]^{P'_n} = \sum_{\ell(F) \leq \min(n,c)} (\rho_n^{F^*})^{P'_n} \otimes \rho_c^F$$

Then, by considering ρ_c^F as a $\mathrm{GL}_{\underline{c}}$-module, we have the following decomposition

$$\begin{aligned} \rho_c^F &= \sum_{(D_1,\cdots,D_m)} m^F_{(D_1,\cdots,D_m)} \left(\rho_{c_1}^{D_1} \otimes \cdots \otimes \rho_{c_m}^{D_m}\right) \\ &\cong \sum_{(D_1,\cdots,D_m)} \mathrm{Hom}_{\mathrm{GL}_{\underline{c}}}\left(\rho_{c_1}^{D_1} \otimes \cdots \otimes \rho_{c_m}^{D_m}, \rho_c^F\right) \otimes \left(\rho_{c_1}^{D_1} \otimes \cdots \rho_{c_m}^{D_m}\right) \end{aligned}$$

where $m^F_{(D_1,\cdots,D_m)}$ is the multiplicity of the irreducible $\mathrm{GL}_{\underline{c}}$ module $\rho_{c_1}^{D_1} \otimes \cdots \otimes \rho_{c_m}^{D_m}$ appearing in ρ_c^F. By further taking its invariants under the action of $P'_{\underline{c}}$, we have the decomposition of $\mathbb{C}[\mathrm{Mat}_{n,c}]^{P'_n \times P'_{\underline{c}}}$

$$\sum_F \sum_{(D_1,\cdots,D_m)} \mathrm{Hom}_{\mathrm{GL}_{\underline{c}}}\left(\rho_{c_1}^{D_1} \otimes \cdots \rho_{c_m}^{D_m}, \rho_c^F\right) \otimes \left(\rho_{c_1}^{D_1}\right)^{P'_{c_1}} \otimes \cdots \otimes \left(\rho_{c_m}^{D_m}\right)^{P'_{c_m}}$$

By Lemma 4.7, the dimension of the space $\left(\rho_{c_1}^{D_1}\right)^{P'_{c_1}} \otimes \cdots \otimes \left(\rho_{c_m}^{D_m}\right)^{P'_{c_m}}$ is at most 1. Therefore, each graded component, if it is not zero, encodes the branching rule with respect to the restriction of GL_c down to $\mathrm{GL}_{\underline{c}}$. $\square$

4.14. We remark that the algebra $\mathbb{C}[\mathrm{Mat}_{n,c}]^{P'_n}$ can be understood as the multi-homogeneous coordinate algebra of the flag variety

$$Y_{\mathbf{n}} = Fl(n_1, \cdots, n_s; \mathbb{C}^c)$$

and its graded component $\left(\rho_n^F\right)^{P'_n} \otimes \rho_c^F$ for $F = (f_1^{n_1}, \cdots, f_s^{n_s})$ with $f_i = e_i + \cdots + e_s$ is exactly the section space $\Gamma(Y_{\mathbf{n}}, L_{\mathbf{e}})$. See [Fu97, §9]. We remark that the graded components are labeled by F and $(D_1, \cdots, D_m)$ or equivalently by $\mathbf{e}$ and $\mathbf{r}$. In fact, it can be identified with the subspace of $\Gamma(Y_{\mathbf{n}}, L_{\mathbf{e}})$ invariant under $P_{\underline{c}}$ and stable under $P_{\underline{c}}/P'_{\underline{c}}$ with the character $\mu_{\mathbf{r}}$ in (3.3), i.e., $\Gamma\left(Y_{\mathbf{n}}, L_{\mathbf{e}}(\mathbf{r})\right)^{P_{\underline{c}}}$.

4.15. Now, by combining two propositions, we have

Corollary 4.16. *The dimension of the* $(\mu_F, \mu_{D(1)}, \cdots, \mu_{D(m)})$*-homogeneous component for the* $\left(A_n^+ \times A_{\underline{c}}^+\right)$*-graded algebra* $\mathbb{C}[\mathrm{Mat}_{n,c}]^{P'_n \times P'_{\underline{c}}}$ *records simultaneously*

(1) *the multiplicity of the* GL_n *module* ρ_n^F *in the tensor product* $\rho_n^{D_1} \otimes \cdots \otimes \rho_n^{D_m}$,
(2) *the multiplicity of the* $\mathrm{GL}_{\underline{c}}$ *module* $\rho_{c_1}^{D_1} \otimes \cdots \rho_{c_m}^{D_m}$ *in* ρ_c^F.

Proof. In the above propositions, we showed that $\mathbb{C}[\mathrm{Mat}_{n,c}]^{P'_n \times P'_{\underline{c}}}$, as a $\left(A_n^+ \times A_{\underline{c}}^+\right)$-graded algebra, has two different decompositions

$$\sum_{(D_1,\cdots,D_m)} \sum_F \mathrm{Hom}_{\mathrm{GL}_n}\left(\rho_n^F, \rho_n^{D_1} \otimes \cdots \otimes \rho_n^{D_m}\right) \otimes \left(\rho_n^{F^*}\right)^{P'_n}$$

$$\sum_F \sum_{(D_1,\cdots,D_m)} \mathrm{Hom}_{\mathrm{GL}_{\underline{c}}}\left(\rho_{c_1}^{D_1} \otimes \cdots \otimes \rho_{c_m}^{D_m}, \rho_c^F\right) \otimes \left(\rho_{c_1}^{D_1}\right)^{P'_{c_1}} \otimes \cdots \otimes \left(\rho_{c_m}^{D_m}\right)^{P'_{c_m}}$$

By comparing the graded components, we see that the dimension of the following multiplicity spaces should be the same

$$\mathrm{Hom}_{\mathrm{GL}_n}\left(\rho_n^F, \rho_n^{D_1} \otimes \cdots \otimes \rho_n^{D_m}\right);$$
$$\mathrm{Hom}_{\mathrm{GL}_{\underline{c}}}\left(\rho_{c_1}^{D_1} \otimes \cdots \otimes \rho_{c_m}^{D_m}, \rho_c^F\right),$$

which correspond to the branching multiplicities in the statement. $\square$

Remark 4.17. Our proofs for the propositions are the same as the one given in [HL07] where maximal unipotent subgroups are used instead of the commutator subgroups of parabolic subgroups.

Corollary 4.16 shows that the $(P'_n \times P'_{\underline{c}})$-invariant subalgebra of $\mathbb{C}[\mathrm{Mat}_{n,c}]$ encodes two different types of branching rules. With this dual interpretation, $\mathbb{C}[\mathrm{Mat}_{n,c}]^{P'_n \times P'_{\underline{c}}}$ can be called a *reciprocity algebra* in the sense of [HL07, HTW08].

Then from (4.1), the $P'_H \times P'_T$ invariant subalgebra of $\mathbb{C}[\mathrm{Rep}(Q, \mathbf{d})]$ can be realized as the tensor product of reciprocity algebras, and as stated in Theorem 4.9, it encodes two sets of different types of multiplicity spaces simultaneously. Hence, $\mathbb{C}[\mathrm{Rep}(Q, \mathbf{d})]^{P'_H \times P'_T}$ can be considered a *multi-reciprocity algebra*.

Remark 4.18. It is also possible to develop a parallel theory in terms of tails starting from

$$\begin{aligned} \mathbb{C}[\mathrm{Rep}(Q, \mathbf{d})] &= \bigotimes_{t \in T} \bigotimes_{a \in Q_1(t)} \mathbb{C}[\mathrm{Mat}_{d_{h(a)}, d_t}] \\ &= \bigotimes_{t \in T} \mathbb{C}[\mathrm{Mat}_{n_t, d_t}] \end{aligned}$$

where $Q_1(t) = \{a \in Q_1 : t(a) = t\}$ and $n_t = \sum_{a \in Q_1(t)} d_{h(a)}$.

4.19. We note that there is a nice representation theoretic interpretation of the geometric condition (3.4). If the multiplicity of ρ_n^F in the tensor product $\rho_n^D \otimes \rho_n^E$ is positive, then the number of boxes in D and E is equal to the number of boxes in F. For each $h \in H$, by iterating this condition on Young diagrams $D(h, i)$ and $F(h)$ in Theorem 4.9, we obtain the condition: the number of boxes in all $D(h, i)$'s should be equal to the number of boxes in $F(h)$.

To be more precise, let $F(h)$ and $D(h, i)$ be given as

$$\begin{aligned} F(h) &= ((e_1 + \cdots + e_s)^{n_1}, (e_2 + \cdots + e_s)^{n_2}, \cdots, e_s^{n_s}); \\ D(h, i) &= ((r_{i,1} + \cdots + r_{i,s_i})^{k_{i,1}}, (r_{i,2} + \cdots + r_{i,s_i})^{k_{i,2}}, \cdots, r_{i,s_i}^{k_{i,s_i}}) \end{aligned}$$

for each i. Then to ensure that the multiplicity of $\rho_{d_h}^{F(h)}$ in the tensor product

$$\rho_{d_h}^{D(h,1)} \otimes \cdots \otimes \rho_{d_h}^{D(h,m(h))}$$

of GL_{d_h} representations in Theorem 4.9 to be non-zero, we need

$$\sum_{1\le i\le m}\sum_{1\le j\le s_i} k_{i,j}(r_{i,j}+\cdots+r_{i,s_i}) = \sum_{1\le i\le s} n_i(e_i+\cdots+e_s).$$

For the whole algebra $\mathbb{C}[\mathrm{Rep}(Q,\mathbf{d})]$, by repeating this over all $h\in H$ and adding all of them together, we obtain the same condition we imposed for the linearizations (3.4).

4.20. As a special case, let us consider the star quiver given in §2.9 with the dimension vector

$$\mathbf{d} = (n,1,\cdots,1)\in\mathbb{N}^{m+1}$$

for $n\le m$. In this case, the invariant sections can be explicitly described in terms of the combinatorics of Young tableaux.

For a partition $n = n_1+\cdots+n_s$, let us consider its corresponding parabolic subgroup P_n of GL_n and the torus $T_m = (\mathbb{C}^*)^m$. The section space $\Gamma(Y_{\mathbf{n}}, L_{\mathbf{e}})$ of $Y_{\mathbf{n}} = Fl(n_1,\cdots,n_s;\mathbb{C}^m)$ can be identified with the summand $(\rho_n^{F^*})^{P'_n}\otimes\rho_m^F$ of $\mathbb{C}[\mathrm{Mat}_{n,m}]^{P'_n\times 1}$ for

$$\begin{aligned} F &= (f_1^{n_1}, f_2^{n_2},\cdots,f_s^{n_s}) \\ &= ((e_1+\cdots+e_s)^{n_1},(e_2+\cdots+e_s)^{n_2},\cdots,e_s^{n_s}). \end{aligned}$$

The space $(\rho_n^{F^*})^{P'_n}\otimes\rho_m^F\cong\rho_m^F$ consists of $f\in\mathbb{C}[\mathrm{Mat}_{n,m}]$ which are invariant under P'_n and eigenvectors under P_n/P'_n with weight

$$\mu_F(\tau_1,\cdots,\tau_s) = \prod \tau_i^{n_i f_i}.$$

They can be identified with the products of determinants or semistandard Young tableaux of diagram F having entries from $\{1,\cdots,m\}$.

Since the T_m-eigenspace of ρ_m^F with weight $\mu_{\mathbf{r}}(t_1,\cdots,t_m) = \prod t_i^{r_i}$ is exactly the space spanned by the weight vectors of ρ_m^F with weight $\mu_{\mathbf{r}}$, the elements in the T_m-invariant section space $\Gamma(Y_{\mathbf{n}}, L_{\mathbf{e}}(\mathbf{r}))^{T_m}$ can be realized as the space spanned by the products of determinants identifiable with semistandard Young tableaux of diagram F and content $\mathbf{r} = (r_1,\cdots,r_m)$. For further detail, see, for example, [Fu97, §9] and [Ki08].

5. The Transfer Principle

5.1. The so-called *transfer principle* is a useful tool to study quotients by non-reductive groups.

Theorem 5.2. ([Gro97, Theorem 9.1]) For a linear algebraic group G, let Z be a rational G-module and a subgroup H of G be acting on $\mathbb{C}[G]$ by right translation. If Z is a $\mathbb{C}$-algebra, then there is an algebra isomorphism

$$Z \cong (\mathbb{C}[G] \otimes Z)^G$$

which is H-equivariant. In particular, we have

$$Z^H \cong \left(\mathbb{C}[G]^H \otimes Z\right)^G.$$

Let us compare our results with the transfer principle. We recall that Λ_k^+ denotes the semigroup of polynomial dominant weights for GL_k.

Proposition 5.3. *As $(\Lambda_n^+ \times \Lambda_m^+)$-graded algebras, we have*

$$\mathbb{C}[\mathrm{Mat}_{n,m}]^{P'_n \times P'_m} \cong \left(\mathbb{C}[\mathrm{GL}_m]^{1 \times P'_m} \otimes \mathbb{C}[\mathrm{Mat}_{n,m}]^{P'_n \times 1}\right)^{\mathrm{GL}_m}$$

Proof. From Lemma 4.7, the left hand side decomposes as

$$\sum_F (\rho_n^{F^*})^{P'_n} \otimes (\rho_m^{\lambda})^{P'_m}$$

where the sum runs over all Young diagrams F of length not more than $\min(n, m)$.

For the right hand side, we note that the ring of regular functions over GL_m decomposes as

$$\mathbb{C}[\mathrm{GL}_m] = \sum_\lambda \rho_m^{\lambda^*} \otimes \rho_m^{\lambda}$$

over all rational dominant weights λ for GL_m (cf. [GW09, Theorem 4.2.7]). By combining this with Lemma 4.7, the right hand side decomposes as

$$\begin{aligned} & \sum_{\lambda,F} \left(\rho_m^{\lambda^*} \otimes (\rho_m^{\lambda})^{P'_m} \otimes (\rho_n^{F^*})^{P'_n} \otimes \rho_m^{F}\right)^{\mathrm{GL}_m} \\ = \ & \sum_{\lambda,F} (\rho_n^{F^*})^{P'_n} \otimes (\rho_m^{\lambda})^{P'_m} \otimes \left(\rho_m^{\lambda^*} \otimes \rho_m^{F}\right)^{\mathrm{GL}_m} \end{aligned}$$

Since the dimension of the invariant space $\left(\rho_m^{\lambda^*} \otimes \rho_m^F\right)^{GL_m}$ is not more than 1 and it is exactly 1 when $\lambda = F$. This shows that two graded algebras are isomorphic. $\square$

For a fence quiver Q with dimension vector d, as given in §3 and §4, by iterating the above proposition, it is easy to see that

Corollary 5.4. *As* $(A_H^+ \times A_T^+)$*-graded algebras, we have*

$$\mathbb{C}[\mathrm{Rep}(Q, \mathbf{d})]^{P'_H \times P'_T} \cong \left(\mathbb{C}[G_T]^{1 \times P'_T} \otimes \mathbb{C}[\mathrm{Rep}(Q, \mathbf{d})]^{P'_H \times 1}\right)^{G_T}$$

where $A_H^+ = \prod_h A_{d_h}^+$ *and* $A_T^+ = \prod_t A_{d_t}^+$.

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